

HÖLDER ESTIMATES FOR NONLOCAL-DIFFUSION EQUATIONS WITH DRIFTS

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ABSTRACT. We study a class of nonlocal-diffusion equations with drifts, and derive a priori Φ -Hölder estimate for the solutions by using a purely probabilistic argument, where Φ is an intrinsic scaling function for the equation.

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1. INTRODUCTION

It is well-known that a priori Hölder estimates play a key role in the study of non-linear equations. There is a wealth literature on this for differential operators. In this paper, we study a priori Hölder estimates of parabolic functions for a large class of time-dependent non-local operators with (time-dependent) gradient drifts on \mathbb{R}^d

$$\mathcal{L}_t^b u = \mathcal{L}_t u + b_t \cdot \nabla u.$$

Here $b_t(x)$ is an \mathbb{R}^d -valued measurable function on $[0, \infty) \times \mathbb{R}^d$ and \mathcal{L}_t is a time-dependent purely non-local operator given by

$$\mathcal{L}_t u(x) = \int_{\mathbb{R}^d} [u(x+z) + u(x-z) - 2u(x)] \kappa_t(x, z) dz, \quad (1.1)$$

where $\kappa_t(x, z)$ is symmetric in z (i.e. $\kappa_t(x, z) = \kappa_t(x, -z)$) and satisfies

$$\sup_{t,x} \int_{\mathbb{R}^d} (1 \wedge |z|^2) \kappa_t(x, z) dz \leq C_1, \quad (1.2)$$

and for some regularly varying function ϕ with index $\alpha \in [0, 2]$ (see Definition 3.1 below),

$$\frac{c_1}{\phi(|z|)|z|^d} \leq \kappa_t(x, z) \leq \frac{c_1^{-1}}{\phi(|z|)|z|^d} \quad \text{for } |z| \leq 3. \quad (1.3)$$

Note that under condition (1.2), $\mathcal{L}_t u(x)$ is well-defined and bounded for every $u \in C_b^2(\mathbb{R}^d)$. Since $\kappa_t(x, z)$ is symmetric in z , we can rewrite $\mathcal{L}_t u(x)$ in (1.1) as

$$\begin{aligned} \mathcal{L}_t u(x) &= 2 \text{p.v.} \int_{\mathbb{R}^d} (u(x+z) - u(x)) \kappa_t(x, z) dz \\ &= 2 \int_{\mathbb{R}^d} (u(x+z) - u(x) - \nabla u(x) \cdot z \mathbf{1}_{|z| \leq 1}) \kappa_t(x, z) dz \end{aligned}$$

for $u \in C_b^2(\mathbb{R}^d)$.

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When $\kappa_t(x, z) = c|z|^{-d-\alpha}$ for some suitable constant $c > 0$ and $\alpha \in (0, 2)$, \mathcal{L}_t is just the usual fractional Laplacian $\Delta^{\alpha/2}$. Recently Silvestre [13] proved the following a priori Hölder estimate for the fractional-diffusion equation with drift

$$\partial_t u = \Delta^{\alpha/2} u + b_t \cdot \nabla u. \quad (1.4)$$

There are constants $C > 0$ and $\beta \in (0, 1)$ such that for any classical solution u of equation (1.4), $x, y \in \mathbb{R}^d$ with $|x - y| \leq 1$ and $0 < s \leq t \leq 1$,

$$|u(t, x) - u(s, y)| \leq C \|u\|_{L^\infty([0,1] \times \mathbb{R}^d)} \frac{|x - y|^\beta + |t - s|^{\beta/\alpha}}{t^{\beta/\alpha}}, \quad (1.5)$$

provided b in the Hölder class $C^{1-\alpha}$ when $\alpha \in (0, 1)$, and bounded measurable when $\alpha \in [1, 2)$, where the constants C and β only depend on $d, \alpha, \|b\|_\infty$, as well as on the $(1 - \alpha)$ -Hölder norm of b when $\alpha \in (0, 1)$. See [9] for recent result on local regularity of solutions to

$$\Delta^{\alpha/2} u + b(x) \cdot \nabla u = f$$

in Sobolev spaces with $\alpha \in (0, 1)$.

In the literature, if $\alpha \in (1, 2)$, the equation (1.4) is usually referred to as the *subcritical* case since the fractional Laplacian $\Delta^{\alpha/2}$ is of higher order than that of the gradient part $b_t \cdot \nabla$. For $\alpha = 1$, it is called the *critical* case since the fractional Laplacian has the same order as the first order gradient term. For $\alpha \in (0, 1)$, it is known as the *supercritical* case because the fractional Laplacian is of lower order than the drift term, and the drift term can be stronger than the diffusion term in small scales. This explains why one needs b to be Hölder continuous for the above a priori estimate in the supercritical case. It should be noted that the following scaling property plays a crucial role in paper [13]: for $\lambda > 0$, let $u^\lambda(t, x) := \lambda^{-\alpha} u(\lambda^\alpha t, \lambda x)$ and $b^\lambda(t, x) := b(\lambda^\alpha t, \lambda x)$, then u^λ satisfies

$$\partial_t u^\lambda = \Delta^{\alpha/2} u^\lambda + \lambda^{\alpha-1} b^\lambda \cdot \nabla u^\lambda.$$

We mention that the Hölder's estimate (1.5) has been used in the study of well-posedness of multidimensional critical Burger's equations in Zhang [14].

In this paper, we are concerned with the following nonlocal-diffusion equation with drift b :

$$\partial_t u = \mathcal{L}_t^b u = \mathcal{L}_t u + b_t \cdot \nabla u. \quad (1.6)$$

Following [11], we define

$$\Phi(r) := \left(\int_r^2 \frac{ds}{s\phi(s)} \right)^{-1}, \quad r \in (0, 1), \quad (1.7)$$

which is a continuous increasing function. The purpose of introducing this function Φ is to deal with the case when ϕ in (1.3) is a regularly varying function of order 0. It is known (see (3.5) below) that $\lim_{r \rightarrow 0} \frac{\Phi(r)}{\phi(r)} = \alpha$. Thus ϕ and Φ are comparable when $\alpha \in (0, 2]$ so we could use ϕ in place of Φ in this case. The function Φ will be used to measure the modulus of continuity of parabolic functions; see Theorem 1.1. Such a result would be trivial if $\lim_{r \rightarrow 0} \Phi(r) > 0$. Thus without loss of generality, we will assume in this paper that

$$\lim_{r \rightarrow 0} \Phi(r) = 0. \quad (1.8)$$

Note that

$$\lim_{r \rightarrow 0} \Phi(r) = 0 \iff \int_0^2 \frac{ds}{s\phi(s)} = \infty \stackrel{(1.3)}{\iff} \int_{\mathbb{R}^d} \kappa_t(x, z) dz = \infty.$$

In [11], Kassmann and Mimica called Φ an intrinsic scaling function and obtained Φ -Hölder regularity of harmonic functions for the (time-independent) non-local operator \mathcal{L}_0 with $b_0 = 0$ under conditions (1.2), (1.3) and (1.8). When $\kappa_0(x, z)$ satisfies (1.3) for all $z \in \mathbb{R}^d$ and $b_0 = 0$, Hölder regularity of harmonic functions of \mathcal{L}_0 was first established in Bass and Levin [2].

In this paper, we use “:=” as a way of definition. Let $\mathbb{R}^+ := [0, \infty)$, and $C_c^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ the space of smooth functions with compact support in $\mathbb{R}^+ \times \mathbb{R}^d$. A probability measure \mathbf{Q} on the Skorokhod space $\mathbb{D}([0, \infty); \mathbb{R}^+ \times \mathbb{R}^d)$ is said to be a solution to the martingale problem for $(\mathcal{L}_t^b, C_c^\infty(\mathbb{R}^+ \times \mathbb{R}^d))$ with initial value $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^d$ if $\mathbf{Q}(Z_0 = (t, x)) = 1$ and for every $f \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$,

$$M_s^f := f(s+t, X_s) - f(t, X_0) - \int_0^s (\partial_r + \mathcal{L}_{t+r}^b) f(t+r, X_r) dr \quad (1.9)$$

is a \mathbf{Q} -martingale. The martingale problem for $(\mathcal{L}_t^b, C_c^\infty(\mathbb{R}^+ \times \mathbb{R}^d))$ with initial value $(t, x) \in \mathbb{R}^d$ is said to be well-posed if it has a unique solution.

Throughout this paper, we assume conditions (1.2) and (1.3) and (1.8) hold. Here are our main results.

Theorem 1.1. *Assume that the martingale problem for $(\mathcal{L}_t^b, C_c^\infty(\mathbb{R}^+ \times \mathbb{R}^d))$ is well-posed for every initial value $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^d$. Let Φ be defined by (1.7).*

(i) *Suppose that $\liminf_{r \rightarrow 0} r/\Phi(r) = 0$ and for some $C_2 > 0$,*

$$\frac{\Phi(r)}{\Phi(s)} \leq C_2 \frac{r}{s} \quad \text{for } 0 < s \leq r \leq 1, \quad (1.10)$$

and $b_t(x)$ is continuous in x for each $t > 0$ having

$$\|b/(1+|x|)\|_\infty := \sup_{t \in [0,1], x \in \mathbb{R}^d} |b_t(x)/(1+|x|) < \infty$$

with

$$|b_t(x) - b_t(y)| \leq C_b |x - y| / \Phi(|x - y|) \quad \text{for } |x - y| \leq 1 \quad (1.11)$$

for some $C_b > 0$. Then there are constants $\beta \in (0, 1)$ and $\lambda = \lambda(\|b/(1+|x|)\|_\infty) > 0$ such that for any classical solution u of (1.6), and for any $0 < t \leq t_0 \leq 1$ and $|x_0 - x| + \lambda(t_0 - t) \leq \Phi^{-1}(t_0)$,

$$|u(t_0, x_0) - u(t, x)| \leq 16 \|u\|_{L^\infty([0,1] \times \mathbb{R}^d)} t_0^{-\beta} ((t_0 - t) + \Phi(|x_0 - x| + \lambda(t_0 - t)))^\beta. \quad (1.12)$$

(ii) *Suppose $\liminf_{r \rightarrow 0} r/\Phi(r) > 0$ and b is a bounded measurable function on $\mathbb{R}^+ \times \mathbb{R}^d$. Then there is a constant $\beta \in (0, 1)$ such that for any classical solution u of (1.6), and for any $0 < t \leq t_0 \leq 1$ and $|x_0 - x| \leq \Phi^{-1}(t_0)$,*

$$|u(t_0, x_0) - u(t, x)| \leq 16 \|u\|_{L^\infty([0,1] \times \mathbb{R}^d)} t_0^{-\beta} ((t_0 - t) + \Phi(|x_0 - x|))^\beta. \quad (1.13)$$

Remark 1.2. Condition (1.10) is automatically satisfied by (3.1) below when ϕ (also Φ) is regularly varying with $\alpha \in [0, 1)$. Moreover, condition (1.10) is satisfied when Φ is comparable to a convex function Ψ with $\Psi(0) = 0$ as in this case $r/\Psi(r)$ is increasing. The following table gives some examples of regularly varying functions ϕ that satisfy conditions (1.2), (1.3) and (1.8), where the α stands for the index of regularly varying function ϕ , the last column denotes the modulus of continuity in spatial variable x required for \mathbb{R}^d -valued function $b_t(x)$.

Case	α	$\phi(s)$	$\Phi(r)$	b
(i)	0	$\ln \frac{3}{s}$	$\asymp (\ln \ln \frac{3}{r})^{-1}$	$s \ln \ln \frac{3}{s}$
(i)	0	1	$(\ln \frac{2}{r})^{-1}$	$s \ln \frac{2}{s}$
(i)	0	$(\ln \frac{3}{s})^{-1}$	$\asymp (\ln \frac{3}{r})^{-2}$	$s(\ln \frac{3}{s})^2$
(i)	$(0, 1)$	s^α	$\asymp r^\alpha$	$s^{1-\alpha}$
(i)	1	$s(\ln \frac{3}{s})^\beta, \beta > 0$	$\asymp r(\ln \frac{3}{r})^\beta$	$(\ln \frac{3}{s})^{-\beta}$
(ii)	1	$s(\ln \frac{3}{s})^\beta, \beta \leq 0$	$\asymp r(\ln \frac{3}{r})^\beta$	1
(ii)	$(1, 2)$	s^α	$\asymp r^\alpha$	1
(ii)	2	$s^2(\ln \frac{3}{s})^\beta, \beta > 1$	$\asymp r^2(\ln \frac{3}{r})^\beta$	1

Remark 1.3. When $\phi(r) = r^\alpha$ with $\alpha \in (0, 2)$ (and so $\Phi(r)$ is comparable to r^α), condition $\liminf_{r \rightarrow 0} r/\Phi(r) = 0$ corresponds precisely to the supercritical case $0 < \alpha < 1$, while $\liminf_{r \rightarrow 0} r/\Phi(r) > 0$ corresponds to $1 \leq \alpha < 2$. So Theorem 1.1 contains the main results of Silvestre [13] as a particular case. Unlike [13], in the supercritical case (i), we do not need to assume b is bounded. Moreover, in this paper we can not only deal with more general but also time-dependent nonlocal operator \mathcal{L}_t^b . In particular, taking $b = 0$, we obtain a priori Φ -Hölder estimates for parabolic functions of time-dependent non-local operator \mathcal{L}_t . It contains as a special case a priori Φ -Hölder estimates for harmonic functions $u(x)$ of non-local operator \mathcal{L}_0 , which is the main result of Kassmann and Mimica [11] when $b_0 = 0$. \square

Remark 1.4. In this paper, we concentrate on a priori Hölder estimates for parabolic functions. For results on the well-posedness of the martingale problems for \mathcal{L}_t^b , we refer the reader to [1, 7, 8] and the references therein. We remark here that if $\kappa_t(x, z)$ is independent of (t, x) , $\mathcal{L}_t = \mathcal{L}_0$ is the generator of a Lévy process Y . When $b_t(x)$ is uniformly Lipschitz in x , it is easy to show that for every initial data $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^d$, stochastic differential equation

$$dX_s^t = dY_s + b_{t+s}(X_s^t)ds \quad (1.14)$$

has a unique strong solution X^t with $X_0^t = x$. (This can be done as follows. For each $\omega \in \Omega$, ODE $dZ_s^t = b_{t+s}(Z_s^t + Y_s(\omega))ds$ with $Z_0^t(\omega) = x$ has a unique solution. Then $X_s^t := Z_s^t + Y_t$ is the unique solution to SDE (1.14) with $X_0^t = x$.) Hence in this case, the martingale problem for $(\mathcal{L}, C_c^\infty(\mathbb{R}^+ \times \mathbb{R}^d))$ is well-posed and so Theorem 1.1 is applicable. It is important to note that the Φ -Hölder estimate in Theorem 1.1 does not depend on the Lipschitz constant of $b_t(x)$.

When the martingale problem for $(\mathcal{L}_t^b, C_c^\infty(\mathbb{R}^+ \times \mathbb{R}^d))$ is well-posed for every initial value $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^d$, there is a space-time Hunt process $Z_s = (V_0 + s, X_s)$ having $\partial_s + \mathcal{L}_{V_0+s}^v$ as its infinitesimal generator. For any bounded classical solution u of (1.6), by Itô's formula, $u(s + t, X_s)$ is a $\mathbf{P}_{(t,x)}$ -martingale for every $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^d$. This is the only property of u we used in the proof of Theorem 1.1. Hence the conclusion of Theorem 1.1 holds for any bounded function u on $\mathbb{R}^+ \times \mathbb{R}^d$ such that $u(s + t, X_s)$ is a $\mathbf{P}_{(t,x)}$ -martingale for every $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^d$. \square

The approach of this paper is purely probabilistic. Our tool is the time-inhomogeneous strong Markov process X determined by the solution of the martingale problem for $(\mathcal{L}_t^b, C_c^\infty(\mathbb{R}^+ \times \mathbb{R}^d))$. In Section 2, we prove an abstract result on Hölder's continuity in terms of certain estimates

on exiting and hitting probabilities, which is motivated by the approaches in [2, 5]. This probabilistic approach has its origin in Krylov and Safanov [12] for diffusion processes associated with second order non-divergence form differential operators. In Section 3, we prove our main results by verifying the abstract conditions. The Lévy system of the strong Markov process X plays a key role in establishing these exiting and hitting probabilities.

2. AN ABSTRACT CRITERION FOR HÖLDER'S REGULARITY

Let Ω be the space of càdlàg functions from $\mathbb{R}^+ = [0, \infty)$ to \mathbb{R}^d , which is endowed with the Skorokhod topology. Let $X_s(\omega) = \omega_s$ be the coordinate process over Ω . Define the space-time process

$$Z_s := (V_s, X_s), \quad V_s := V_0 + s.$$

Let $\{\mathcal{F}_s^0; s \geq 0\}$ be the natural filtration generated by X . Suppose $\{\mathbb{P}_{(t,x)}; t \geq 0, x \in \mathbb{R}^d\}$ is a family of probability measures over $(\Omega, \mathcal{F}_\infty^0)$ so that $Z = (\Omega, \mathcal{F}_\infty^0, \mathcal{F}_s^0, Z_s, \mathbb{P}_{(t,x)})$ is a time-homogeneous strong Markov processes with state space $\mathbb{R}^+ \times \mathbb{R}^d$ with

$$\mathbb{P}_{(t,x)}(Z_0 = (t, x)) = 1.$$

Denote by $\{\mathcal{F}_s; s \geq 0\}$ the minimal augmented filtration of Z . Note that under $\mathbb{P}_{(t,x)}$, $\{X_s^t := X_{s+t}; s \geq 0\}$ is a possibly time-inhomogeneous strong Markov process with

$$\mathbb{P}_{(t,x)}(X_s^t = x, s \in [0, t]) = 1.$$

For a Borel set $A \subset \mathbb{R}^+ \times \mathbb{R}^d$, denote by σ_A, τ_A the hitting time and exit time of A , i.e.,

$$\sigma_A := \inf\{s \geq 0 : Z_s \in A\}, \quad \tau_A := \inf\{s \geq 0 : Z_s \notin A\}.$$

Definition 2.1. A nonnegative Borel measurable function $u(t, x)$ on $\mathbb{R}^+ \times \mathbb{R}^d$ is called Z -harmonic (or simply parabolic) in a relatively open subset D of $\mathbb{R}^+ \times \mathbb{R}^d$ if for each relatively compact open subset $A \subset D$ and every $(t, x) \in A$,

$$u(t, x) = \mathbb{E}_{(t,x)}[u(Z_{\tau_A})]. \quad (2.1)$$

Remark 2.2. Condition (2.1) is equivalent to the following. For any (\mathcal{F}_s) -stopping time τ ,

$$u(t, x) = \mathbb{E}_{(t,x)}[u(Z_{\tau \wedge \tau_A})]. \quad (2.2)$$

Indeed, assume that (2.1) holds for any $(t, x) \in A$. In view of $Z_\tau \in A$ on $\{\tau < \tau_A\}$, we have

$$\mathbf{1}_{\tau < \tau_A} u(Z_\tau) = \mathbf{1}_{\tau < \tau_A} \mathbb{E}_{Z_\tau}[u(Z_{\tau_A})].$$

Let $\{\theta_t; t \geq 0\}$ be the usual shift operators on Ω . By the strong Markov property, we have

$$\begin{aligned} \mathbb{E}_{(t,x)}[\mathbf{1}_{\tau < \tau_A} u(Z_\tau)] &= \mathbb{E}_{(t,x)}[\mathbf{1}_{\tau < \tau_A} \mathbb{E}_{Z_\tau}[u(Z_{\tau_A})]] = \mathbb{E}_{(t,x)}[\mathbf{1}_{\tau < \tau_A} \mathbb{E}_{(t,x)}[u(Z_{\tau_A} \circ \theta_\tau) | \mathcal{F}_\tau]] \\ &= \mathbb{E}_{(t,x)}[\mathbf{1}_{\tau < \tau_A} u(Z_{\tau_A} \circ \theta_\tau)] \quad \text{since } \{\tau < \tau_A\} \in \mathcal{F}_\tau. \end{aligned}$$

Since $\tau_A = \tau + \tau_A \circ \theta_\tau$ on $\{\tau < \tau_A\}$ and $Z_{\tau_A} \circ \theta_\tau = Z_{\tau + \tau_A \circ \theta_\tau}$, we obtain

$$\mathbb{E}_{(t,x)}[\mathbf{1}_{\tau < \tau_A} u(Z_\tau)] = \mathbb{E}_{(t,x)}[\mathbf{1}_{\tau < \tau_A} u(Z_{\tau_A})],$$

which implies that

$$\mathbb{E}_{(t,x)}[u(Z_{\tau \wedge \tau_A})] = \mathbb{E}_{(t,x)}[\mathbf{1}_{\tau < \tau_A} u(Z_\tau)] + \mathbb{E}_{(t,x)}[\mathbf{1}_{\tau \geq \tau_A} u(Z_{\tau_A})] = \mathbb{E}_{(t,x)}[u(Z_{\tau_A})] = u(t, x).$$

Let $\Phi : (0, 2) \rightarrow [0, \infty)$ be a continuous and strictly increasing function with $\Phi(1) = 1$. Write for $r > 0$,

$$B(r) := \{z \in \mathbb{R}^d : |z| < r\} \quad \text{and} \quad Q(r) := [0, \Phi(r)) \times B(r).$$

Define

$$\varphi_a(r) := \Phi^{-1}(a\Phi(r)) \text{ for } a > 0, \quad D_a(r) := Q(\varphi_a(r)) \setminus Q(\varphi_{\sqrt{a}}(r)) \text{ for } a > 1. \quad (2.3)$$

Notice that

$$\varphi_1(r) = r \quad \text{and} \quad a \mapsto \varphi_a(r) \text{ is strictly increasing.}$$

We make the following assumptions:

(H₁) There exist constants $C_3, C_4 \geq 1$ such that for each $a \geq C_4$ and $r, R \in (0, 1)$ with $\varphi_a(r) \leq R$,

$$\sup_{(t_0, x_0) \in Q(r)} \mathbb{P}_{(t_0, x_0)}(X_{\tau_{Q(r)}} \notin B(R)) \leq C_3 \frac{\Phi(r)}{\Phi(R)}. \quad (2.4)$$

(H₂) There is an increasing sequence of positive numbers $\{a_k; k \geq 1\} \subset (1, \infty)$ with $\lim_{k \rightarrow \infty} a_k = \infty$ such that for every $a \in \{a_k; k \geq 1\}$, there exists a constant $\gamma_a \in (0, a]$ so that for each $r \in (0, 1)$ with $\varphi_a(r) \leq 1$, there is a radon measure μ_r over $D_a(r)$ such that for any compact subset $K \subset D_a(r)$ with $\mu_r(K) \geq \frac{1}{3}\mu_r(D_a(r))$,

$$\inf_{(t_0, x_0) \in Q(r)} \mathbb{P}_{(t_0, x_0)}(\sigma_K < \tau_{Q(\varphi_a(r))}) \geq \frac{\gamma_a}{a}, \quad (2.5)$$

and $\lim_{k \rightarrow \infty} \gamma_{a_k} = \infty$.

Remark 2.3. If X_s is a continuous process, then **(H₁)** is automatically satisfied.

Theorem 2.4. Under **(H₁)** and **(H₂)**, there exists a constant $\beta \in (0, 1)$, which only depends on C_3, C_4 and γ_a , such that for each $r \in (0, 1)$, every bounded measurable function u on $[0, 1] \times \mathbb{R}^d$ that is parabolic in $Q(r)$,

$$|u(t, x) - u(0, 0)| \leq 8 \left(\frac{t \vee \Phi(|x|)}{\Phi(r)} \right)^\beta \|u\|_{L^\infty([0, \Phi(r)] \times \mathbb{R}^d)} \quad \text{for } (t, x) \in Q(r). \quad (2.6)$$

Proof. Our proof is adapted from Chen and Kumagai [5, Theorem 4.14]. Fix $r \in (0, 1)$. Without loss of generality, we may assume $0 \leq u \leq 1$ on $[0, \Phi(r)] \times \mathbb{R}^d$. Otherwise, instead of u , we may consider

$$\tilde{u}_t(x) = \frac{u_t(x) - \inf_{(s, y) \in [0, \Phi(r)] \times \mathbb{R}^d} u_s(y)}{\sup_{(s, y) \in [0, \Phi(r)] \times \mathbb{R}^d} u_s(y) - \inf_{(s, y) \in [0, \Phi(r)] \times \mathbb{R}^d} u_s(y)}.$$

(i) Define for $n \in \mathbb{N}$,

$$r_n := \varphi_{a^{1-n}}(r), \quad s_n := 2b^{1-n},$$

where $a > 1$ from $\{a_k; k \geq 1\}$ and $b \in (1, 2)$ to be determined below. Observe that $\Phi(r_n) = a\Phi(r_{n+1})$. Clearly,

$$\varphi_a(r_{n+1}) = r_n \text{ and } r_n \downarrow 0, \quad s_n \downarrow 0.$$

For simplicity of notation, we write

$$Q_n := Q(r_n), \quad M_n := \sup_{Q_n} u, \quad m_n := \inf_{Q_n} u.$$

We are going to prove that the oscillation of u over Q_k

$$\text{osc}_{Q_k} u := M_k - m_k \leq s_k, \quad k \in \mathbb{N}. \quad (2.7)$$

If this is proven, then (2.6) follows. In fact, for any $(t, x) \in \mathcal{Q}_1$, there is an $n \in \mathbb{N}$ such that

$$(t, x) \in \mathcal{Q}_n \setminus \mathcal{Q}_{n+1},$$

which means that

$$\Phi(r_{n+1}) \leq t < \Phi(r_n) = a\Phi(r_{n+1}) \quad \text{or} \quad r_{n+1} \leq |x| < r_n.$$

In this case, we have

$$|u(t, x) - u(0, 0)| \leq M_n - m_n \leq s_n = 2ba^{-n \ln b / \ln a} \leq 2b \left(\frac{t \vee \Phi(|x|)}{\Phi(r)} \right)^{\frac{\ln b}{\ln a}},$$

and (2.6) follows with $\beta = \ln b / \ln a$.

(ii) We now prove (2.7) by an inductive argument. First of all, clearly,

$$M_1 - m_1 \leq 1 \leq s_1 = 2, \quad M_2 - m_2 \leq 1 \leq s_2 = 2/b.$$

Next suppose that $M_k - m_k \leq s_k$ for all $k = 1, \dots, n$. Define

$$A := \left\{ z \in D_a(r_{n+1}) : u(z) \leq \frac{m_n + M_n}{2} \right\}.$$

By considering $1 - u$ instead of u if necessary, we may assume that

$$\mu_{r_{n+1}}(A) \geq \frac{1}{2} \mu_{r_{n+1}}(D_a(r_{n+1})),$$

where $\mu_{r_{n+1}}$ is given in (\mathbf{H}_2) . (Note here we are interested in the oscillation $\text{osc}_{\mathcal{Q}_k} u = M_k - m_k$ not on the exact values of M_k and m_k .) Since $\mu_{r_{n+1}}$ is regular, there is a compact subset $K \subset A$ such that

$$\mu_{r_{n+1}}(K) \geq \frac{1}{3} \mu_{r_{n+1}}(D_a(r_{n+1})). \quad (2.8)$$

For any $\varepsilon > 0$, let us choose $z_1, z_2 \in \mathcal{Q}_{n+1}$ so that

$$u(z_1) \leq m_{n+1} + \varepsilon, \quad u(z_2) \geq M_{n+1} - \varepsilon.$$

If one can show

$$u(z_2) - u(z_1) \leq s_{n+1}, \quad (2.9)$$

then

$$M_{n+1} - m_{n+1} - 2\varepsilon \leq s_{n+1} \Rightarrow M_{n+1} - m_{n+1} \leq s_{n+1},$$

and (2.7) is thus proven.

(iii) Now, we show (2.9). Since $z_2 \in \mathcal{Q}_{n+1} \subset \mathcal{Q}_n$, if we define $\tau_n := \tau_{\mathcal{Q}_n}$, then by (2.2) we have

$$\begin{aligned} u(z_2) - u(z_1) &= \mathbb{E}_{z_2} \left[u(Z_{\tau_n \wedge \sigma_K}) - u(z_1) \right] \\ &= \left(\mathbb{E}_{z_2} \left[u(Z_{\sigma_K}) - u(z_1); \sigma_K < \tau_n \right] + \mathbb{E}_{z_2} \left[u(Z_{\tau_n}) - u(z_1); \sigma_K \geq \tau_n; Z_{\tau_n} \in \mathcal{Q}_{n-1} \right] \right) \\ &\quad + \mathbb{E}_{z_2} \left[u(Z_{\tau_n}) - u(z_1); \sigma_K \geq \tau_n, Z_{\tau_n} \notin \mathcal{Q}_{n-1} \right] \\ &=: I_1 + I_2. \end{aligned} \quad (2.10)$$

For I_1 , since $u(z_1) \geq m_{n+1} \geq m_n \geq m_{n-1}$, by the inductive hypothesis we have

$$\begin{aligned} I_1 &\leq \left(\frac{m_n + M_n}{2} - m_n \right) \mathbb{P}_{z_2}(\sigma_K < \tau_n) + (M_{n-1} - m_{n-1}) \mathbb{P}_{z_2}(\sigma_K \geq \tau_n) \\ &\leq \frac{s_n}{2} \mathbb{P}_{z_2}(\sigma_K < \tau_n) + s_{n-1} (1 - \mathbb{P}_{z_2}(\sigma_K < \tau_n)) \end{aligned}$$

$$\leq s_{n-1}(1 - \mathbb{P}_{z_2}(\sigma_K < \tau_n)/2) \leq s_{n+1}b^2(1 - \gamma_a/(2a)), \quad (2.11)$$

where the last step is due to (2.8) and (\mathbf{H}_2) . For I_2 , we similarly have

$$\begin{aligned} I_2 &= \sum_{i=1}^{n-2} \mathbb{E}_{z_2} \left[u(Z_{\tau_n}) - u(z_1); \sigma_K \geq \tau_n, Z_{\tau_n} \in Q_{n-i-1} \setminus Q_{n-i} \right] \\ &\quad + \mathbb{E}_{z_2} \left[u(Z_{\tau_n}) - u(z_1); \sigma_K \geq \tau_n, Z_{\tau_n} \notin Q_1 \right] \\ &\leq \sum_{i=1}^{n-2} s_{n-i-1} \mathbb{P}_{z_2}(Z_{\tau_n} \notin Q_{n-i}) + \mathbb{P}_{z_2}(Z_{\tau_n} \notin Q_1). \end{aligned}$$

Noticing that

$$\mathbb{P}_{z_2}(Z_{\tau_n} \notin Q_{n-i}) = \mathbb{P}_{z_2}(X_{\tau_{Q(r_n)}} \notin B(r_{n-i})),$$

by (\mathbf{H}_1) , we further have for $a > \max\{C_4, b\}$,

$$I_2 \leq C_3 \sum_{i=1}^{n-2} s_{n-i-1} \frac{\Phi(r_n)}{\Phi(r_{n-i})} + C_3 \frac{\Phi(r_n)}{\Phi(r_1)} = 2C_3 b^{2-n} \sum_{i=1}^{n-2} (b/a)^i + C_3 a^{1-n} \leq s_{n+1} b^2 \left(\frac{C_3 b}{a-b} + \frac{C_3}{2a} \right),$$

which, together with (2.10) and (2.11), yields that

$$u(z_2) - u(z_1) \leq s_{n+1} b^2 \left(1 - \frac{\gamma_a}{2a} + \frac{C_3 b}{a-b} + \frac{C_3}{2a} \right) \leq s_{n+1} b^2 \left(1 - \frac{\gamma_a}{3a} \right) \leq s_{n+1}$$

provided we take $a = a_k$ large enough and b close to 1 as $\lim_{k \rightarrow \infty} \gamma_{a_k} = \infty$. This completes the proof. \square

3. PROOF OF THEOREM 1.1

We first recall the definition and properties of regularly varying functions.

Definition 3.1. A measurable and positive function $\phi : (0, 2) \rightarrow (0, \infty)$ is said to vary regularly at zero with index $\alpha \in \mathbb{R}$ if for every $\lambda > 0$,

$$\lim_{r \rightarrow 0} \frac{\phi(\lambda r)}{\phi(r)} = \lambda^\alpha.$$

We call such ϕ a regularly varying function. All regularly varying functions with index α is denoted by \mathcal{R}_α .

We list some properties of $\phi \in \mathcal{R}_\alpha$ for later use (cf. [3, p.25-28] and [11]).

Proposition 3.2. Let $\alpha \geq 0$ and $\phi \in \mathcal{R}_\alpha$ be bounded away from 0 and ∞ on any compact subset of $(0, 2)$. For any $\delta > 0$, there is a constant $C_5 = C_5(\delta) \geq 1$ such that for all $r, s \in (0, 1]$,

$$\frac{\phi(r)}{\phi(s)} \leq C_5 \max \left\{ \left(\frac{r}{s} \right)^{\alpha+\delta}, \left(\frac{r}{s} \right)^{\alpha-\delta} \right\}, \quad (3.1)$$

and for any $\beta > \alpha - 1$,

$$\lim_{r \rightarrow 0} \frac{\phi(r)}{r^{\beta+1}} \int_0^r \frac{s^\beta}{\phi(s)} ds = (\beta - \alpha + 1)^{-1}, \quad (3.2)$$

$$\lim_{r \rightarrow 0} r^{\beta+1-\alpha} \phi(r) \int_r^2 \frac{1}{s^{\beta+2-\alpha} \phi(s)} ds = (\beta - \alpha + 1)^{-1}. \quad (3.3)$$

Moreover, if we define

$$\Phi(r) := \left(\int_r^2 \frac{1}{\phi(s)s} ds \right)^{-1}, \quad (3.4)$$

then $\Phi \in \mathcal{R}_\alpha$ and

$$\lim_{r \rightarrow 0} \frac{\Phi(r)}{\phi(r)} = \alpha. \quad (3.5)$$

In particular, (3.1) and (3.2) also hold for Φ , and for some $C_6 > 1$,

$$\phi(2s) \leq C_6 \phi(s) \quad \text{and} \quad \Phi(2s) \leq C_6 \Phi(s) \quad \text{for } s \in (0, 1/2). \quad (3.6)$$

We now return to the setting in Section 1. By normalizing the function ϕ in (1.3) by a constant multiple, we may and do assume the scale function Φ defined by (1.7) has the property that $\Phi(1) = 1$. Consider the nonlocal operator \mathcal{L}_t^b in (1.6). We assume

(MP) The martingale problem for $(\mathcal{L}_t^b, C_c^\infty(\mathbb{R}^+ \times \mathbb{R}^d))$ is well-posed for every initial value $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^d$.

Denote by $\mathbb{P}_{(t,x)}$ the law of the unique solution to the martingale problem for $(\mathcal{L}_t^b, C_c^\infty(\mathbb{R}^+ \times \mathbb{R}^d))$ with initial value $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^d$. By [10, Theorems 4.3.12 and 4.4.2]), $Z_s = (V_0 + s, X_s)$ is a Hunt process with $\mathbb{P}_{(t,x)}(V_0 = t \text{ and } X_0 = x) = 1$ and so it has a Lévy system that describes the jumps of Z . By a similar argument as that for [4, Theorem 2.6], we have the following.

Theorem 3.3. Assume **(MP)** holds. Then for any $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^d$ and any non-negative measurable function f on $\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d$ vanishing on $\{(s, x, y) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d : x = y\}$ and (\mathcal{F}_t) -stopping time T ,

$$\mathbb{E}_{(t,x)} \left[\sum_{s \leq T} f(s, X_{s-}, X_s) \right] = \mathbb{E}_{(t,x)} \left[\int_0^T \left(\int_{\mathbb{R}^d} f(s, X_s, y) \kappa_{s+t}(X_s, y - X_s) dy \right) ds \right]. \quad (3.7)$$

Next we prove the following estimate, which implies **(H₁)**.

Lemma 3.4. Let C_6 be as in (3.6). Under (1.2), (1.3) and **(MP)**, there is a constant $C_7 \geq 1$ such that for all $a \geq C_6$ and $r, R \in (0, 1)$ with $\varphi_a(r) \leq R$,

$$\sup_{(t_0, x_0) \in Q(r)} \mathbb{P}_{(t_0, x_0)}(X_{\tau_{Q(r)}} \notin B(R)) \leq C_7 \frac{\Phi(r)}{\Phi(R)},$$

where Φ is defined by (3.4).

Proof. For simplicity of notation, we write $z = (t_0, x_0)$. Note that $r < \varphi_a(r)$ so we have by formula (3.7),

$$\begin{aligned} \mathbb{P}_z(X_{\tau_{Q(r)}} \notin B(R)) &= \mathbb{E}_z \left(\sum_{0 < s \leq \tau_{Q(r)}} \mathbf{1}_{\{X_{s-} \in B(r), X_s \in B(R)^c\}} \right) = \mathbb{E}_z \int_0^{\tau_{Q(r)}} \int_{B(R)^c} \kappa_{s+t}(X_s, X_s - y) dy ds \\ &= \mathbb{E}_z \int_0^{\tau_{Q(r)}} \int_{B(2) \cap B(R)^c} \kappa_{s+t}(X_s, X_s - y) dy ds + \mathbb{E}_z \int_0^{\tau_{Q(r)}} \int_{B(2)^c} \kappa_{s+t}(X_s, X_s - y) dy ds =: I_1 + I_2. \end{aligned}$$

By (3.6), we have $\varphi_a(r) \geq 2r$ for $a \geq C_6$, which implies that

$$|x - y| \geq |y| - |x| \geq |y|/2 \quad \text{for } x \in B(r) \text{ and } y \in B(R)^c \subset B(\varphi_a(r))^c.$$

For I_1 , by (1.3) and (3.1) we have

$$\begin{aligned} I_1 &\leq \mathbb{E}_z \int_0^{\tau_{Q(r)}} \int_{B(2) \cap B(R)^c} \frac{c_1^{-1}}{\phi(|X_s - y|)|X_s - y|^d} dy ds \\ &\leq C \mathbb{E}_z \tau_{Q(r)} \int_{B(2) \cap B(R)^c} \frac{dy}{\phi(|y|)|y|^d} \leq C \mathbb{E}_z \tau_{Q(r)} / \Phi(R). \end{aligned}$$

On the other hand, by (1.2) we clearly have

$$I_2 \leq \mathbb{E}_z \int_0^{\tau_{Q(r)}} \int_{B(1)^c} \kappa_{s+t}(X_s, y) dy ds \leq C_1 \mathbb{E}_z \tau_{Q(r)}.$$

Hence, by (1.8),

$$\mathbb{P}_z(X_{\tau_{Q(r)}} \notin B(R)) \leq \mathbb{E}_z \tau_{Q(r)} (C_1 + C/\Phi(R)) \leq C_7 \mathbb{E}_z \tau_{Q(r)} / \Phi(R),$$

which yields the desired estimate by $\tau_{Q(r)} \leq \Phi(r)$. \square

Before verifying (\mathbf{H}_2) , we need the following lemma.

Lemma 3.5. *Let Φ be defined by (3.4). Suppose that one of the following conditions holds:*

(i) $\liminf_{r \rightarrow 0} r/\Phi(r) = 0$ and for some $C_2 > 0$,

$$\frac{\Phi(r)}{\Phi(s)} \leq C_2 \frac{r}{s}, \quad 0 < s \leq r \leq 1, \quad (3.8)$$

and for some $C_b > 0$,

$$|b_t(x)| \leq C_b |x| / \Phi(|x|), \quad |x| \leq 1. \quad (3.9)$$

(ii) $\liminf_{r \rightarrow 0} r/\Phi(r) > 0$ and b is bounded measurable.

Then there exists a constant $C_8 \geq 1$ such that for all $r \in (0, 1)$, $x_0 \in B(r)$ and $t_0 \in [0, 1]$,

$$\mathbb{P}_{(t_0, x_0)}(\tau_{B(x_0, r)} < t) \leq \frac{C_8 t}{\Phi(r)}, \quad t > 0. \quad (3.10)$$

In particular, for any $a \geq C_6^2 > 1$ and $r \in (0, 1)$ with $\varphi_a(r) \leq 1$,

$$\sup_{x_0 \in B(r)} \mathbb{P}_{(t_0, x_0)}(\tau_{Q(\varphi_a(r))} < \Phi(r)) \leq \frac{C_8}{\sqrt{a}}, \quad (3.11)$$

where C_6 is the positive constant in (3.6).

Proof. Given $f \in C_b^2(\mathbb{R}^d)$ with $f(0) = 0$ and $f(x) = 1$ for $|x| \geq 1$, set

$$f_r(x) := f((x - x_0)/r), \quad r > 0.$$

By the optional stopping theorem,

$$\mathbb{P}_{(t_0, x_0)}(\tau_{B(x_0, r)} < t) \leq \mathbb{E}_{(t_0, x_0)} f_r(X_{\tau_{B(x_0, r)} \wedge t}) = \mathbb{E}_{(t_0, x_0)} \int_0^{\tau_{B(x_0, r)} \wedge t} \mathcal{L}_{s+t_0}^b f_r(X_s) ds. \quad (3.12)$$

On the other hand, by the definition of \mathcal{L}_s^b and (1.3), we have

$$\begin{aligned} |\mathcal{L}_s^b f_r(x)| &= \left| \int_{\mathbb{R}^d} (f_r(x+z) + f_r(x-z) - 2f_r(x)) \kappa_s(x, z) dz + b_s(x) \cdot \nabla f_r(x) \right| \\ &\leq C \int_{|z| \leq r} \frac{\|\nabla^2 f_r\|_\infty}{\phi(|z|)|z|^{d-2}} dz + C \int_{1 \geq |z| \geq r} \frac{\|f_r\|_\infty}{\phi(|z|)|z|^d} dz + \|f_r\|_\infty \int_{|z| \geq 1} \kappa_s(x, z) dz + \|\nabla f_r\|_\infty |b_s(x)| \end{aligned}$$

$$\begin{aligned}
&\leq \frac{C}{r^2} \int_0^r \frac{s ds}{\phi(s)} + C \int_r^1 \frac{ds}{\phi(s)s} + C + \frac{C|b_s(x)|}{r} \\
&\leq \frac{C}{\phi(r)} + \frac{C}{\Phi(r)} + C + \frac{C|b_s(x)|}{r} \quad \text{by (3.2) and (3.3).}
\end{aligned}$$

Substituting this into (3.12) and using (3.5) and (1.8), we obtain

$$\mathbb{P}_{(t_0, x_0)}(\tau_{B(x_0, r)} < t) \leq \frac{Ct}{\Phi(r)} + \mathbb{E}_{(t_0, x_0)} \int_0^{\tau_{B(x_0, r)} \wedge t} \frac{C|b_{s+t_0}(X_s)|}{r} ds. \quad (3.13)$$

In case (i), since

$$|x| \leq |x - x_0| + |x_0| \leq 2r \text{ for } x \in B(x_0, r) \text{ and } x_0 \in B(r),$$

(3.10) follows by (3.13) and

$$\frac{|b_{s+t_0}(x)|}{r} \stackrel{(3.9)}{\leq} \frac{C_b|x|}{r\Phi(|x|)} \stackrel{(3.8)}{\leq} \frac{2C_bC_2}{\Phi(2r)} \leq \frac{C}{\Phi(r)}.$$

In case (ii), (3.10) follows by (3.13) and $|b_{s+t_0}(X_s)| \leq \|b\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)}$ as well as $\frac{1}{r} \leq \frac{C}{\Phi(r)}$.

On the other hand, by (3.6), we have for any $a \geq C_6^2$,

$$\Phi(2\varphi_{\sqrt{a}}(r)) \leq C_6 \sqrt{a} \Phi(r) \leq a \Phi(r),$$

which implies that for $x_0 \in B(r)$ and $x \in B(x_0, \varphi_{\sqrt{a}}(r))$,

$$|x| \leq |x - x_0| + |x_0| \leq \varphi_{\sqrt{a}}(r) + r \leq 2\varphi_{\sqrt{a}}(r) \leq \varphi_a(r).$$

Hence $B(x_0, \varphi_{\sqrt{a}}(r)) \subset B(\varphi_a(r))$ and

$$\mathbb{P}_{(t_0, x_0)}(\tau_{Q(\varphi_a(r))} < \Phi(r)) \leq \mathbb{P}_{(t_0, x_0)}(\tau_{B(x_0, \varphi_{\sqrt{a}}(r))} < \Phi(r)) \stackrel{(3.10)}{\leq} \frac{C_8 \Phi(r)}{\Phi(\varphi_{\sqrt{a}}(r))} = \frac{C_8}{\sqrt{a}}.$$

The proof is complete. \square

For $a > 1$ and $r \in (0, 1)$ with $\varphi_a(r) \leq 1$, let $D_a(r)$ be defined by (2.3). Define a measure

$$\mu_r(A) := \int_0^{\Phi(r)} \int_{\mathbb{R}^d} \frac{1_A(s, y) \Phi(|y|)}{\phi(|y|)|y|^d} dy ds, \quad A \subset D_a(r).$$

The above definition of μ_r arises naturally when estimating the lower bound of $\mathbb{P}_{(t_0, x_0)}(\sigma_K < \tau_{Q(\varphi_a(r))})$ using the Léve system of Z ; see (3.16) below. Clearly, we have

$$\begin{aligned}
\mu_r(D_a(r)) &= \Phi(r) \int_{\varphi_{\sqrt{a}}(r) \leq |y| \leq \varphi_a(r)} \frac{\Phi(|y|)}{\phi(|y|)|y|^d} dy = \omega_d \Phi(r) \int_{\varphi_{\sqrt{a}}(r)}^{\varphi_a(r)} \frac{\Phi(s)}{\phi(s)s} ds \\
&= \omega_d \Phi(r) \int_{\varphi_{\sqrt{a}}(r)}^{\varphi_a(r)} \frac{1}{\Phi(s)} d\Phi(s) = \frac{1}{2} \omega_d \Phi(r) \ln a,
\end{aligned} \quad (3.14)$$

where ω_d is the sphere area of the unit ball.

Lemma 3.6. *Suppose (1.2), (1.3), (MP) and the assumptions of Lemma 3.5 hold. There exist $a_0 \geq 1$ and $c_2 \in (0, 1)$ such that for each $a \geq a_0$ and $r \in (0, 1)$ with $\varphi_a(r) \leq 1$, and any compact subset $K \subset D_a(r)$ with $\mu_r(K) > \frac{1}{3}\mu_r(D_a(r))$,*

$$\inf_{(t_0, x_0) \in Q(r)} \mathbb{P}_{(t_0, x_0)}(\sigma_K < \tau_{Q(\varphi_a(r))}) \geq c_2 \frac{\ln a}{a}. \quad (3.15)$$

In particular, condition (\mathbf{H}_2) holds.

Proof. Notice that

$$\{Z_{\tau_{Q(\varphi_{\sqrt{a}(r)})}} \in K\} \subset \{\sigma_K < \tau_{Q(\varphi_a(r))}\}.$$

It suffices to prove that there are $a_0 \geq 1$ and $c_2 \in (0, 1)$ such that for all $a \geq a_0$ and any $(t_0, x_0) \in Q(r)$,

$$\mathbb{P}_{(t_0, x_0)}(Z_{\tau_{Q(\varphi_{\sqrt{a}(r)})}} \in K) \geq c_2 \frac{\ln a}{a}.$$

As $\mu_r(\partial Q(\varphi_{\sqrt{a}(r)})) = 0$, by taking a suitable subset of K if needed, we may assume without loss of generality that $K \cap \partial Q(\varphi_{\sqrt{a}(r)}) = \emptyset$. Then

$$\mathbf{1}_K(Z_{\tau_{Q(\varphi_{\sqrt{a}(r)})}}) = \sum_{0 < s \leq \tau_{Q(\varphi_{\sqrt{a}(r)})}} \mathbf{1}_{X_{s-} \neq X_s} \mathbf{1}_K(Z_s).$$

Hence by formula (3.7) and (1.3), we have

$$\begin{aligned} \mathbb{P}_{(t_0, x_0)}(Z_{\tau_{Q(\varphi_{\sqrt{a}(r)})}} \in K) &= \mathbb{E}_{(t_0, x_0)} \int_0^{\tau_{Q(\varphi_{\sqrt{a}(r)})}} \int_{\mathbb{R}^d} \mathbf{1}_K(s, y) \kappa_s(X_s, X_s - y) dy ds \\ &\geq c_1 \mathbb{E}_{(t_0, x_0)} \int_0^{\tau_{Q(\varphi_{\sqrt{a}(r)})}} \int_{\mathbb{R}^d} \frac{\mathbf{1}_K(s, y)}{\phi(|X_s - y|) |X_s - y|^d} dy ds. \end{aligned}$$

Since $|x - y| \leq 2|y|$ for $x \in B(\varphi_{\sqrt{a}(r)})$ and $y \notin B(\varphi_{\sqrt{a}(r)})$, by (3.1) and definition of μ_r , we have

$$\begin{aligned} \mathbb{P}_{(t_0, x_0)}(Z_{\tau_{Q(\varphi_{\sqrt{a}(r)})}} \in K) &\geq c_3 \mathbb{E}_{(t_0, x_0)} \left(\int_0^{\tau_{Q(\varphi_{\sqrt{a}(r)})}} \int_{\mathbb{R}^d} \frac{\mathbf{1}_K(s, y)}{\phi(|y|) |y|^d} dy ds \right) \\ &= c_3 \mathbb{E}_{(t_0, x_0)} \left(\int_0^{\tau_{Q(\varphi_{\sqrt{a}(r)})}} \int_{\mathbb{R}^d} \frac{\mathbf{1}_K(s, y)}{\Phi(|y|)} \mu_r(dy, ds) \right) \\ &\geq \frac{c_3 \mu_r(K)}{\Phi(\varphi_a(r))} \mathbb{P}_{(t_0, x_0)}(\tau_{Q(\varphi_{\sqrt{a}(r)})} \geq \Phi(r)), \end{aligned} \tag{3.16}$$

where the last inequality is due to $y \in D_a(r)$ and the increasing of Φ . Lastly, by $\mu_r(K) \geq \frac{1}{3} \mu_r(D_a(r))$, (3.14) and (3.11), we obtain that for $a \geq C_6^2 \vee 4C_8^2 =: a_0$,

$$\begin{aligned} \mathbb{P}_{(t_0, x_0)}(Z_{\tau_{Q(\varphi_{\sqrt{a}(r)})}} \in K) &\geq \frac{c_3 \omega_d \ln a}{6a} (1 - \mathbb{P}_{(t_0, x_0)}(\tau_{Q(\varphi_{\sqrt{a}(r)})} < \Phi(r))) \\ &\geq \frac{c_3 \omega_d \ln a}{6a} \left(1 - \frac{C_8}{\sqrt{a}} \right) \geq \frac{c_3 \omega_d \ln a}{12a}. \end{aligned}$$

The proof is completed by taking $c_2 = \frac{c_3 \omega_d}{12}$. \square

We can now present the

Proof of Theorem 1.1. Fix $t_0 > 0$ and $x_0 \in \mathbb{R}^d$.

(i) In this case, by assumption $b_t(x)$ is continuous in x and $b_t(x) \leq C(1 + |x|)$ for all $t > 0$ and $x \in \mathbb{R}^d$. Thus, by the theory of ODE, the following ODE admits at least one solution y_t for $t \in [0, t_0]$:

$$\dot{y}_t = -b_{t_0-t}(x_0 + y_t), \quad y_0 = 0.$$

Define

$$w(t, x) := u(t_0, x_0) - u(t_0 - t, x_0 + x + y_t)$$

and

$$\tilde{b}_t(x) := b_t(x + x_0 + y_t) - b_t(x_0 + y_t).$$

Then

$$\partial_t w + \mathcal{L}_{t_0-t}^b w = 0, \quad t \in [0, t_0).$$

Notice that by (1.11),

$$|\tilde{b}_t(x)| \leq C_b |x| / \Phi(|x|) \quad \text{for } |x| \leq 1.$$

By Lemmas 3.4-3.6 and Theorem 2.4, we have

$$|w(t, x)| = |w(t, x) - w(0, 0)| \leq 8 \left(\frac{t \vee \Phi(|x|)}{t_0} \right)^\beta \|w\|_{L^\infty([0, t_0] \times \mathbb{R}^d)} \quad \text{for } (t, x) \in Q(\Phi^{-1}(t_0)).$$

By making the change of variables $t_0 - t = t'$ and $x_0 + x + y_t = x'$, and noticing that

$$|y_t| \leq \lambda t \quad \text{for some } \lambda = \lambda(\|b/(1 + |x|)\|_\infty) > 0,$$

we obtain the desired estimate (1.12).

(ii) In this case, define

$$w(t, x) := u(t_0, x_0) - u(t_0 - t, x_0 + x).$$

Just as above, one can conclude that (1.13) holds. \square

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